AP Calculus – 4.1 Notes Interpreting Derivatives & Particle Motion (Position-Velocity-Acceleration) 1

Key

Recall:

Slope between two points: $\frac{\Delta y}{\Delta x}$ or $\frac{\Delta \text{dependent}}{\Delta \text{independent}}$

Units for the Derivative:

The derivative of $f(x)$ is $\frac{\text{unit for } f}{\text{unit for } x}$

If $f'(x) > 0$, then $f(x)$ is increasing If $f'(x) < 0$, then $f(x)$ is decreasing

1. Mr. Sullivan wants Mr. Brust to finish creating his packets in Algebra 2. The number of packets Mr. Brust has completed is modeled by $p(w)$, where w is measured in weeks.
   a. Interpret $p(10) = 1$ in the context of the problem.

   After 10 weeks, Mr. Brust has completed 1 packet

   b. Interpret $p'(39) = 4$ in the context of the problem.

   at the 39th week, he is making packets at a rate of 4 packets per week

2. The rate at which Mr. Kelly is buying baseball cards per year is modeled by $b(t)$, where t is measured in years.
   a. Interpret $b(3) = 150$ in the context of the problem.

   On the 3rd year, Mr. Kelly is buying cards at a rate of 150 cards per year.

   b. Interpret $b'(4) = 10$ in the context of the problem.

   On the 4th year, the rate at which Mr. Kelly is buying baseball cards is increasing at a rate of 10 cards per year per year

Practice problems:

For each problem, a differentiable function is given along with a definition of the variables. Interpret the values in the context of the problem.

1. The percentage grade a student receives on a test, is modeled by $G(t)$ where t is the number of hours spent studying for the test. Interpret $G'(1) = 3$.

   At 1 hour of studying, the grade will improve at a rate of 3% per hour.

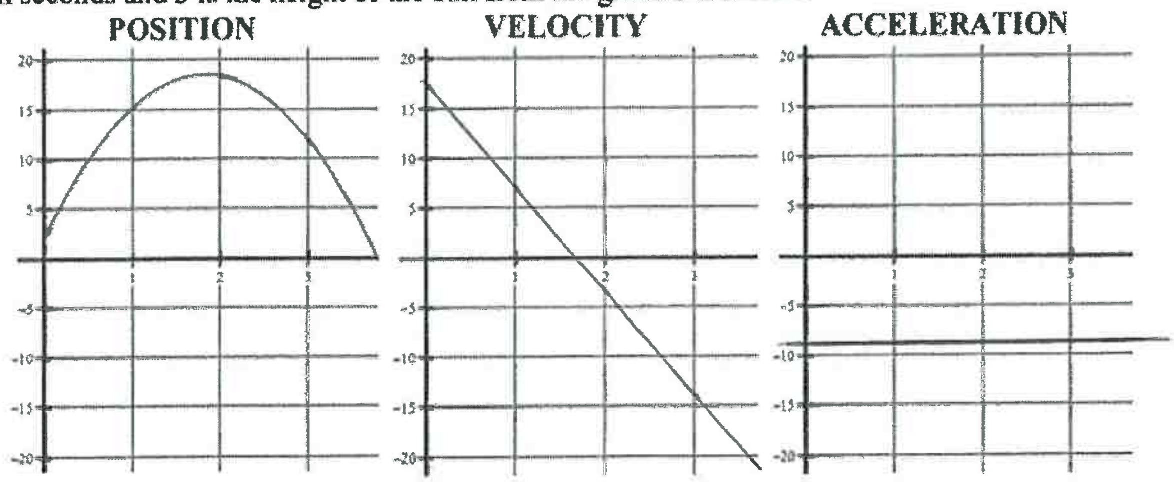
2. Mr. Bean rides his motor scooter to work some days. His distance from home can be modeled by $d(t)$ meters where t is measured in minutes. Interpret $d'(15) = 650$.

   At 15 mins, the distance from home is increasing at a rate of 650 meters per minute.

2

Particle Motion – Position – Velocity – Acceleration (PVA)

Mr. Brust is playing catch with his best friend, himself. He throws a tennis ball straight up into the air. The height of the ball is modeled by $s(t) = -4.9t^2 + 18t + 2$ where t is time in seconds and s is the height of the ball from the ground in meters.



Position function: $s(t)$

Velocity function: $v(t) = s'(t)$

Acceleration function: $a(t) = v'(t) = s''(t)$

Velocity = Rate of Change of Position

$v(t) < 0$ means the particle is moving left (x-axis) or down (y-axis)

$v(t) > 0$ means the particle is moving right (x-axis)

$v(t) = 0$ means the particle is at rest

Average velocity = $\frac{\text{change in position}}{\text{change in time}} \rightarrow \frac{s(b)-s(a)}{b-a}$

(Avg Rate of change)

Speed = |velocity|

Speeding Up or Slowing Down?

If velocity and acceleration have the **same** sign, the particle is speeding up

If velocity and acceleration have **different** signs, the particle is slowing down

t	-5	1	2	4
$v(t)$	3	-2	1	-1
$a(t)$	-4	7	0.1	-1
Conclusion	slowing down	slowing down	speeding up	speeding up

Displacement: The net change in position

$\boxed{s(b) - s(a)}$

Rate of change of velocity

Is velocity increasing or decreasing at $t=1$? Since $a(1) > 0$, velocity is increasing at $t=1$

Particle Motion from an equation.

The position (x-coordinate) of a particle moving on the x-axis is modeled by the function

$$x(t) = t^3 - 4t^2 + 3 \text{ for } t \geq 0,$$

Where x is measured in cm and t is measured in minutes.

1. Find the displacement of the particle during the first 2 minutes.

$x(0) = 3$ | $x(2) - x(0) = -5 - 3 = \boxed{-8\text{cm}}$

$x(2) = -5$

This means that after 2 mins, particle is 8cm to the left of where it started.

2. Find the average velocity of the particle during the first 2 minutes.

$x(0) = 3$ | Avg. ROC | $\frac{x(2)-x(0)}{2-0} \to \frac{-5-3}{2-0} = \frac{-8}{2} = \boxed{-4\text{ cm/min}}$

$x(2) = -5$

3. Find the velocity of the particle when $t = 4$.

*instantaneous velocity | $x'(t) = v(t) = 3t^2 - 8t$ | $v(4) = 3(4)^2 - 32 = \boxed{16\text{ cm/min}}$

4. Find the acceleration of the particle when $t = 4$.

$x''(t) = v'(t) = a(t) = 6t - 8$ $\quad$ $a(4) = 6(4) - 8 = \boxed{16\text{ cm/min}^2}$

5. Is the particle speeding up or slowing down at $t = 4$? Justify.

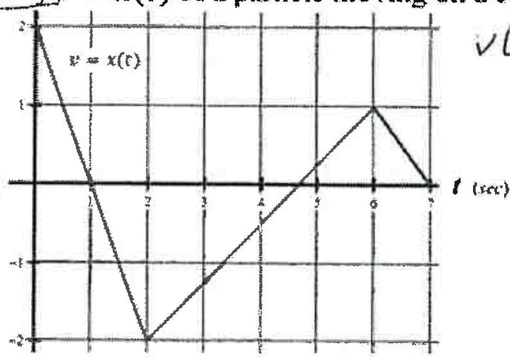
particle is speeding up at $t=4$ since $v(4) > 0$ and $a(4) > 0$ (velocity and acceleration have same signs)

Particle Motion from a graph

The figure shows the velocity $v = x(t)$ of a particle moving on a coordinate line.

positive velocity

negative velocity



v(t) graph

6. When is the particle moving right? Justify.

particle moves right on interval $(0,1), (4.8, 7)$ since $v(t) > 0$

7. When is the particle moving left? Justify.

particle moves left on interval $(1, 4.8)$ since $v(t) < 0$

8. When is the particle's acceleration Positive? Negative? Zero?

*$a(t) > 0$ when $v(t)$ have positive slope | particle's acceleration is positive on $(2,6)$ since $v'(t) > 0$. particle's acceleration is negative on $(0,2)$ and $(6,7)$ since $v'(t) < 0$. Accelerration is never zero since $v'(t)$ is never equal to zero.

9. When does the particle have the greatest speed?

*speed = |velocity| | particle has greatest speed at $t = 0$ (2 units/sec) and $t = 2$ (-2 units/sec)

10. When is the particle speeding up? Justify.

particle speeding up on $(5,6)$ since $v(t) > 0$ and $a(t) > 0$ (same signs)
also speeding up on $(1,2)$ since $v(t) < 0$ and $a(t) < 0$ (same signs)

11. When is the particle slowing down? Justify.

slowing down $(0,1)$, $(2, 4.8)$, and $(6,7)$ since $v(t)$ and $a(t)$ have opposite signs.

Key 5

4.1 AP Practice Problems (p.270)

1. An equation of the tangent line to the graph of $f(x) = 2x^3 - 3x + 6$ at the point $(0, f(0))$ is

(A) $y = -3x - 6$ (B) $y = 3x + 6$

(C) $y = -3x - 3$ $\boxed{\text{(D)}}$ $y = -3x + 6$

point: $f(0) = 0 - 0 + 6 = 6$

$f'(x) = 6x^2 - 3$

$f'(0) = 6(0)^2 - 3 = -3$

point: $(0, 6)$

slope: $m = -3$

$y - 6 = -3(x - 0)$

$\boxed{y = -3x + 6}$

2. If $y = f(x) = x^2 + \ln x$, what is the rate of change of f at 2?

$\boxed{\text{(A)}}$ $\frac{9}{2}$ (B) $\frac{9}{4}$ (C) $\frac{7}{2}$ (D) $\frac{7}{4}$

$y'(x) = 2x + \frac{1}{x}$

$y'(2) = 2(2) + \frac{1}{2} = \boxed{4.5 \text{ or } 9/2}$

3. The position function s of an object in rectilinear motion is $s(t) = \frac{t^3}{6} - \frac{2t^2}{3} + 4t - 1$. At $t = 3$, the object is

$\boxed{\text{(A)}}$ accelerating. (B) decelerating.

(C) neither accelerating nor decelerating. (D) stopped.

$s(t) = \frac{1}{6}t^3 - \frac{2}{3}t^2 + 4t - 1$

$v(t) = \frac{1}{6} \cdot 3t^2 - \frac{2}{3} \cdot 2t + 4$

$v(t) = \frac{1}{2}t^2 - \frac{4}{3}t + 4$

$a(t) = 2 \cdot \frac{1}{2}t - \frac{4}{3}$

$a(t) = t - 4/3$

$\boxed{a(3) = 3 - \frac{4}{3} = \frac{5}{3} > 0}$

Since $a(3) > 0$, the object is accelerating.

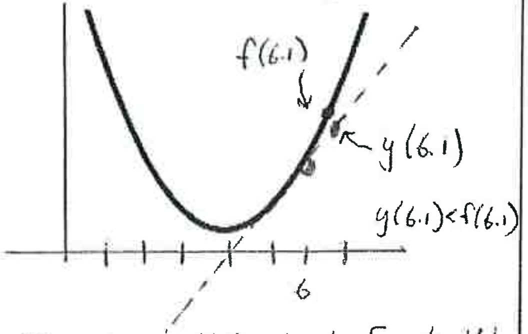
AP Calculus – 4.2 Notes Linear Approximation and rates of change other than motion

6

Key

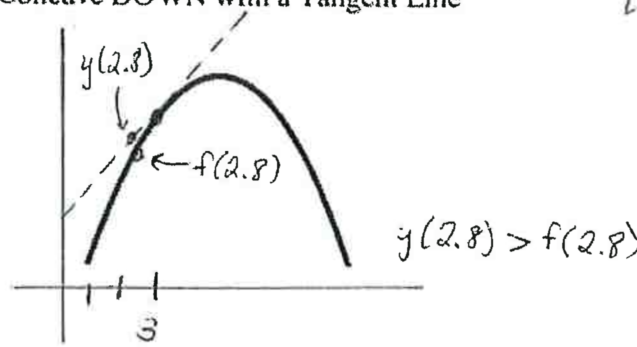
The tangent line of the function $f(x)$ at $x = a$ can give you an approximate value of $f(x)$ for points close to $x = a$.

Concave UP with a Tangent Line



Tangent Line Underestimates Function Value

Concave DOWN with a Tangent Line



Tangent Line Overestimates Function Value

1. f is concave up on its domain and $f(4) = 5$ and $f'(4) = 3$.
   a. What is the estimate for $f(3.8)$ using the local linear approximation for f at $x = 4$?

*Create Tangent line using the given information:

point: $(4,5)$ | $y-5=3(x-4)$ | $y=3(x-4)+5$ | $y(3.8)=3(3.8-4)+5=\boxed{4.4}$

slope: $m=3$

   b. Is it an underestimate or overestimate? Explain.

Since $f(x)$ is concave up, the linear approximation will produce an underapproximation

2. The function $f(x) = 5x - 2x^3 - 2$ is concave down at $x = 1$?.
   a. Find the tangent line of f at $x = 1$.

$f(1)=5(1)-2(1)^3-2=1$ | $f'(1)=5-6(1)^2=-1$ | point: $(1,1)$ | $\boxed{y-1=-1(x-1)}$

$f'(x)=5-6x^2$ | slope: $m=-1$

   b. What is the estimate for $f(1.1)$ using the local linear approximation for f at $x = 1$?

$y=-1(x-1)+1$ | $y(1.1)=-1(1.1-1)+1=\boxed{0.9}$

   c. Is it an underestimate or overestimate? Explain.

Overestimation since $f(x)$ is concave down

*differential equation is just a fancy way of saying a derivative equation

3. Consider the differential equation $\frac{dy}{dx} = e^y(2x^2 - 5x)$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 0$.
   a. Write an equation for the line tangent to the graph of f at the point (2,0).

$\left.\frac{dy}{dx}\right|_{(2,0)} = e^0[2(2)^2-5(2)]=-2$ | point: $(2,0)$ | slope: $m=-2$ | $\boxed{y-0=-2(x-2)}$

   b. Use the tangent line to approximate $f(2.2)$.

$\boxed{y(2.2)=-2(2.2-2)=-0.4}$

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Rates of Change other than Motion:

Increasing or Decreasing?

To know if something is increasing or decreasing, check the Sign of it's derivative

Height is *increasing* if $h'(t) > 0$

Velocity is *decreasing* if $V'(t) < 0$

Recall: Derivative on a calculator.

Find $f'(3)$ if $f(x) = 5^{\sin x}$

$f'(3) = -1.9996$

Is the function already a rate of change?

- If $f(x)$ is the bunny population after x years, than what is $f'(x)$?

  The rate of change of the bunny population per year at year "x"

- If $f(x)$ is the rate at which a bunny population increases (bunnies per year), than what is $f'(x)$?

  The rate at which the bunny population is increasing or decreasing (bunny/yr^2)

Rate of Change from a Table

t (years)	0	10	20	30
$P(t)$ (people)	100	120	150	200

Estimate $P'(15)$

$\frac{P(20)-P(10)}{20-10} \rightarrow \frac{150-120}{20-10} = \boxed{3 \text{ people/yr}}$

Estimate $P'(20)$

$\frac{p(30)-p(20)}{30-20}$ or $\frac{p(20)-p(10)}{20-10}$

Practice Problems:

1. A store is having a 12-hour sale. The total number of shoppers who have entered the store t hours after the sale begins is modeled by the function E defined by $E(t) = 0.3t^4 - 14t^3 + 110t^2$ for $0 \leq t \leq 12$. At what rate are shoppers entering the store 5 hours after the start of the sale?

   $E'(5) = 200$ shoppers per hour.

2. The function $t = f(P)$ models the time, in days, for a small pond to evaporate as a function of the size P of the pond, measured in liters. What are the units for $f''(P)$?

   $\boxed{\text{days/Liters}^2}$

Key 10

4.2 AP Practice Problems (p.281)

1. Let f be a function for which $f(2)=6$ and $f'(2)=-3$. If the tangent line to the graph of f at 2 is used to approximate a zero of f, then the approximation is

OMIT

(A) 0 $\boxed{\text{(B)}}$ 4 (C) 6 (D) 12

2. For small, positive values of h, $\sqrt[3]{8+h}$ is best approximated by

(A) $4-\frac{h}{12}$ (B) $4+\frac{h}{12}$ (C) $2-\frac{h}{12}$ $\boxed{\text{(D)}}$ $2+\frac{h}{12}$

$y=\sqrt[3]{x}$

$y'=\frac{1}{3}x^{-2/3}$

$y'(8)=\frac{1}{3}\left(\frac{1}{8}\right)^{2/3}=\frac{1}{12}$

point: $y(8)=\sqrt[3]{8}=2$

slope: $y'(8)=\frac{1}{12}$

$y-2=\frac{1}{12}(x-8)$

$y=\frac{1}{12}(x-8)+2$

$y(8+h)=\frac{1}{12}[8+h-8]+2=\boxed{\frac{h}{12}+2}$

3. A linear approximation to $f(x)=x\sin\left(\frac{\pi x}{2}\right)+x^2$ at $x=3$ is

(A) $y=5x+6$ $\boxed{\text{(B)}}$ $y=5x-9$

(C) $y=7x-9$ (D) $y=7x+9$

$f'(x)=\underbrace{1}_{f'}\cdot\underbrace{\sin\left(\frac{\pi}{2}x\right)}_{g}+\underbrace{x}_{f}\cdot\underbrace{\cos\left(\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}}_{g'}+2x$

$f'(3)=\sin\left(\frac{3\pi}{2}\right)+3\cancel{\cos\left(\frac{3\pi}{2}\right)}\cdot\frac{\pi}{2}+2(3)$

$f'(3)=-1+6=5$

$f(3)=3\sin\left(\frac{3\pi}{2}\right)+3^2=6$

point: $(3,6)$

slope: $m=5$

$y-6=5(x-3)$

$y=5x-15+6$

$\boxed{y=5x-9}$

4. Using the tangent line to the graph of $f(x)=xe^x+2$ at 0, the approximate value of $f(-0.3)$ is

(A) 2.3 (B) 1.3 $\boxed{\text{(C)}}$ 1.7 (D) -2.3

$f'(x)=\underbrace{1}_{f'}\cdot\underbrace{e^x}_{g}+\underbrace{x}_{f}\cdot\underbrace{e^x}_{g'}+0$

$f'(0)=e^0+0e^0=1$

$f(0)=0+2=2$

point: $(0,2)$

slope: $m=1$

$y-2=1(x-0)$

$y=x+2$

$y(-0.3)=-0.3+2=\boxed{1.7}$

5. If $f'(x)=2xe^{x^2-1}-3\pi\sin(\pi x)$ and $f(1)=4$, approximate $f(1.03)$ using a linear approximation.

$\boxed{\text{(A)}}$ 4.06 (B) 5.06 (C) 4 (D) 3.94

$f'(1)=2(1)e^0-3\pi\sin\pi$

$f'(1)=2$

point: $(1,4)$

slope: $m=2$

$y-4=2(x-1)$

$y=2(x-1)+4$

$y(1.03)=2(1.03-1)+4$

$\boxed{y(1.03)=4.06}$

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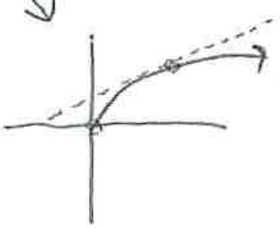
6. The tangent line to the graph of $f(x) = x^3 + 1$ at $x = 1$ is used to approximate $f(x)$ near 1. Which number below is the greatest value of x that results in an error less than or equal to 0.5?

(A) 1.30 $\boxed{\text{(B)}}$ 1.35 (C) 1.40 (D) 1.45

OMIT

7. A linear approximation L is used to approximate $f(x) = \sqrt{x}$, at c, $c > 0$. The approximation

(A) always underestimates the true value of f at c.
$\boxed{\text{(B)}}$ always overestimates the true value of f at c.
(C) sometimes overestimates the true value of f at c.
(D) does not provide enough information to determine whether the true value of f at c is over- or underestimated.



8. Suppose $y = f(x)$ is a differentiable function. The table below gives values of f and f' for select numbers x in the domain of f. Use a linear approximation to approximate $f(3.1)$.

x	-3	0	1	3	5
$f(x)$	4	4	-1	-2	3
$f'(x)$	1	-1	-2	3	4

(A) 4.1 (B) -2.3 (C) 0.1 $\boxed{\text{(D)}}$ -1.7

$f(3) = -2$
$f'(3) = 3$

point: $(3, -2)$
slope: $m = 3$

$y + 2 = 3(x-3)$
$y = 3(x-3) - 2$
$y(3.1) = 3(3.1-3) - 2$

$\boxed{y(3.1) = -1.7}$

Calculus Notes Ch. 4.3 - Related Rates Solution Key (12)

Related Rates: Problems involving finding the rate of change for a variable with respect to time

This is also an application of implicit differentiation: Finding derivatives of variables with respect to time t.

Related Rates Steps:

1. Write what you are given *use units of measurement to help match appropriate variable to values
2. Write what you are trying to find
3. Write an algebraic or geometric equation relating the variables (needs to be in terms of the rates that you are either given or are trying to find)
4. Differentiate equation with respect to time t
5. Substitute and solve

*Important Note: Remember that when the item is getting bigger, the rate is positive
If the item is getting smaller, the rate is negative – regardless of direction

Example 1: The sides of a square are increasing at a rate of 5 cm/min. How fast is the area increasing when the sides measure 15 cm in length?

represents change in Area w/ respect to time

represents change in side length with respect to time

$$A = x^2$$

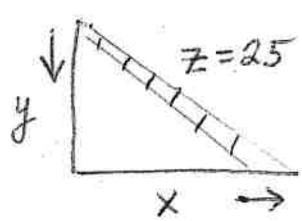
$$\frac{dA}{dt} = 2x\left(\frac{dx}{dt}\right)$$

Given: $\frac{dx}{dt} = 5\ cm/min$ Find $\frac{dA}{dt} = \_\_\_$

$x = 15cm$

$$\frac{dA}{dt} = 2(15)(5) = \boxed{150\ cm^2/min}$$

Example 2: A 25 foot ladder is leaning against a vertical wall. The bottom of the ladder is being pulled horizontally away from the wall at the constant rate of 3 ft / sec. a) How fast is the top of the ladder moving and in what direction when the bottom of the ladder is 15 ft from the wall? b) at what rate is the area changing when the bottom of the ladder is 15 ft from the wall?



$x = 15$

$y = 20$

$z = 25$

$15^2 + y^2 = 25^2$

$y = 20$

$\frac{dx}{dt} = 3$

$\frac{dy}{dt} = \_\_\_$

$\frac{dz}{dt} = 0$

$$x^2 + y^2 = z^2$$

$$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$$

$$2(15)(3) + 2(20)\frac{dy}{dt} = 2(25)(0)$$

$$40\left(\frac{dy}{dt}\right) = -90 \qquad \boxed{\frac{dy}{dt} = -9/4\ ft/s}$$

$$A = \frac{1}{2}xy \qquad f'g + fg'$$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\right)(y) + \frac{1}{2}(x)\left(\frac{dy}{dt}\right)$$

$$\frac{dA}{dt} = \frac{1}{2}(3)(20) + \frac{1}{2}(15)(-9/4)$$

$$= 30 - \frac{135}{8} = \boxed{\frac{105}{8}\ ft^2/s}$$

$$\approx 13.125\ ft^2/s$$

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change in volume (cm^3/s)

Example 3: A spherical balloon is being inflated at a rate of 10 cubic centimeters per second. Find the rate of change of the surface area of the balloon at the moment when the surface area is 64π.

$V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$\frac{dV}{dt} = 10\,cm^3/s$

$\frac{dS}{dt} = \_\_\_\_$

$S = 64\pi$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2\left(\frac{dr}{dt}\right)$

$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

$\frac{dS}{dt} = 8\pi r\left(\frac{dr}{dt}\right)$

$64\pi = 4\pi r^2$

$16 = r^2$

$r = 4$

$10 = 4\pi(4)^2\left(\frac{dr}{dt}\right)$

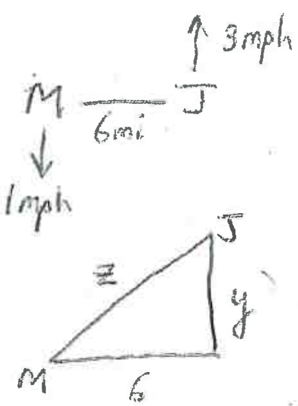
$\frac{10}{64\pi} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{5}{32\pi}$

$\frac{dS}{dt} = 8\pi(4)\left(\frac{5}{32\pi}\right)$

$\boxed{\frac{dS}{dt} = 5\,cm^2/sec}$

Example 4: Joe is standing 6 miles straight east of Moe. If Joe walks straight north at 3 mph while Moe walks straight south at 1 mph, at what rate is the distance between them changing after 2 hours?



$\frac{dy}{dt} = 4$ mph

$y = (4)(2) = 8$ mi

$z = 10$ mi

$6^2 + 8^2 = z^2$

$z = 10$

Find $\frac{dz}{dt} = \_\_\_\_$

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

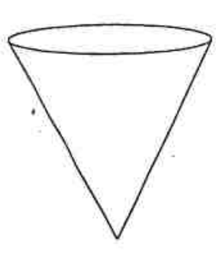
$\cancel{2(6)(0)} + 2(8)(4) = 2(10)\left(\frac{dz}{dt}\right)$

$64 = 20\frac{dz}{dt}$

$\frac{dz}{dt} = \frac{64}{20} = \frac{16}{5} = \boxed{3.2\text{ mph}}$

Example 5: A conical tank (vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

*Use similiar triangles to rewrite equation using less variables.
(find relationship between height and radius, use substitution)

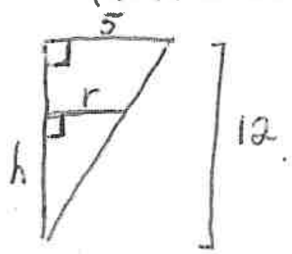


$V = \frac{\pi}{3}r^2h$

$\frac{dV}{dt} = 10\,ft^3/min$

$\frac{dh}{dt} = \_\_\_\_$

$h = 8$ ft.



$\frac{r}{5} = \frac{h}{12}$

$5h = 12r$

$\frac{5}{12}h = r$

*Rewrite Volume in terms of "h"

$V = \frac{\pi}{3}\left(\frac{5h}{12}\right)^2 h = \frac{\pi}{3}\cdot\frac{25h^2}{144}\cdot h = \frac{25\pi}{432}h^3$

$V = \frac{25\pi}{432}h^3$

$\frac{dV}{dt} = \frac{25\pi}{432}\cdot 3h^2\left(\frac{dh}{dt}\right)$

$10 = \frac{25\pi}{432}\cdot 3(8)^2\frac{dh}{dt}$

$10 = \frac{4800\pi}{432}\cdot\frac{dh}{dt}$

$10\cdot\frac{432}{4800\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{9}{10\pi}\text{ ft/min}}$

Key

Related Rates Notes 2 - Similar Triangles and Shadow Problems

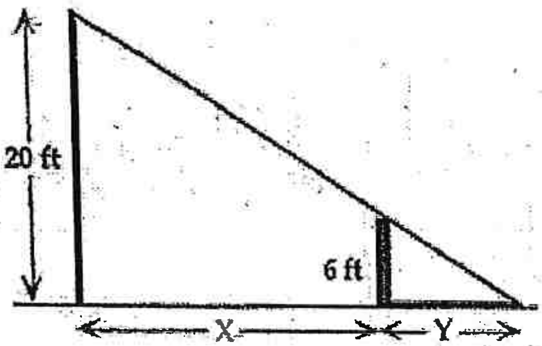
Example 1:

A man who is 6 feet tall is walking away from a lamp post at a rate of 5 feet per ~~minute~~ second.

The lamp post is 20 feet tall. The person casts a shadow on the ground in front of them.

a) How fast is the shadow growing when the person is 30 feet from the lamp post? $\rightarrow \frac{dy}{dt}$

b) How fast is the tip of the shadow moving when the person is 30 ft from the lamp post? $\rightarrow \frac{dx}{dt} + \frac{dy}{dt}$



Notes:

1) $\frac{dx}{dt} = \textit{rate of person walking}$

2) $\frac{dy}{dt} = \textit{rate of change of shadow length}$

3) $\frac{dx}{dt} + \frac{dy}{dt} = \textit{rate of change of tip of shadow}$

$\frac{dx}{dt} = 5 \text{ ft/s}$

$x = 30 \text{ ft}$

$\frac{6}{20} = \frac{y}{x+y}$

$6(x+y) = 20y$

$6x + 6y = 20y$

$6x = 14y$

$6\left(\frac{dx}{dt}\right) = 14\left(\frac{dy}{dt}\right)$

$6(5) = 14\left(\frac{dy}{dt}\right)$

$\frac{30}{14} = \frac{dy}{dt}$

$\frac{15}{7} = \frac{dy}{dt}$

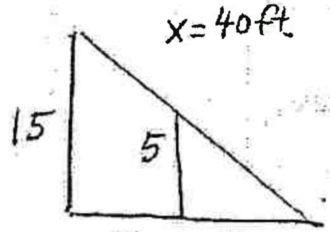
a) $\frac{dy}{dt} = \boxed{\frac{15}{7} \text{ ft/}}\text{min}$

b) $\frac{dx}{dt} + \frac{dy}{dt} = 5 + \frac{15}{7}$

$= \boxed{\frac{50}{7} \text{ or } 7.14 \text{ ft/}}$ min

2. A street light is mounted at the top of a 15 ft pole. A man 5 ft tall walks towards the pole at a rate of 5ft per second. A) How fast is the tip of his shadow moving when he is 40 ft from the pole? B) How fast is the length of the shadow changing when he is 40 ft from the pole?

$x = 40 \text{ ft}$



$\frac{5}{15} = \frac{y}{x+y}$

$\frac{dx}{dt} = -5 \text{ ft/s}$

$\frac{1}{3} = \frac{y}{x+y}$

$3y = x + y$

$2y = x$

$2\left(\frac{dy}{dt}\right) = \frac{dx}{dt}$

$2\left(\frac{dy}{dt}\right) = -5$

$\frac{dy}{dt} = \frac{-5}{2} \text{ ft/s}$

a) $\frac{dx}{dt} + \frac{dy}{dt} = -5 - \frac{5}{2}$

$= \boxed{-7.5 \text{ ft/s}}$

b) $\frac{dy}{dt} = \boxed{\frac{-5}{2} \text{ ft/s} \text{ or } -2.5 \text{ ft/s}}$

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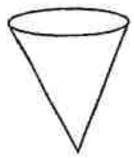
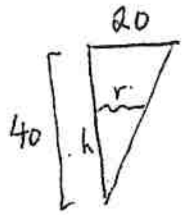
3. A conical tank (vertex down) is 40 feet across the top and 40 feet deep. If water is leaking out of the tank at a rate of 80 cubic feet per minute, find the rate of change of the radius of the water when the water is 8 feet deep. ($V = \frac{1}{3}\pi r^2 h$)





$\frac{dV}{dt} = -80\ ft^3/min$

$h = 8\ ft$

$\frac{dr}{dt} = \_\_\_$

$\frac{r}{20} = \frac{h}{40}$

$20h = 40r$

$h = \frac{40}{20} r$

$h = 2r$, $\underline{\underline{r=4}}$

* Since $h=2r$ and $h=8$, $\underline{\underline{r=4}}$

$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} r^2 (2r)$

$V = \frac{2\pi}{3} r^3$

$\frac{dV}{dt} = \frac{2\pi}{3} \cdot 3r^2 \left(\frac{dr}{dt}\right)$

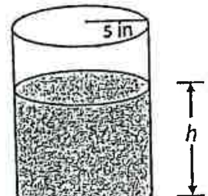
$\frac{dV}{dt} = 2\pi r^2 \left(\frac{dr}{dt}\right)$

$-80 = 2\pi (4)^2 \left(\frac{dr}{dt}\right)$

$\frac{-80}{32\pi} = \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = -\frac{5}{2\pi}\ ft/min}$

4. **2003 AB problem #5**



A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let *h* be the depth of the coffee in the pot, measured in inches, where *h* is a function of time, *t*, measured in seconds. The volume, *V*, of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$.) Find $\frac{dh}{dt}$ as a function of *h*. (This means your answer will contain the variable *h*)

$\frac{dV}{dt} = -5\pi\sqrt{h}$

$r = 5\ in.$

$V = \pi r^2 h$

$V = \pi (5)^2 h$

$V = 25\pi h$

$\frac{dV}{dt} = 25\pi \left(\frac{dh}{dt}\right)$

$-5\pi\sqrt{h} = 25\pi \frac{dh}{dt}$

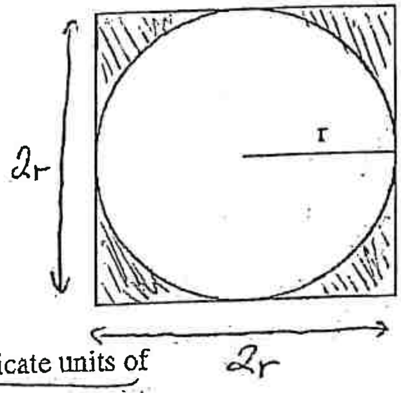
$\frac{-5\pi\sqrt{h}}{25\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{-\sqrt{h}}{5}\ in/sec.}$

AP Calculus AB   Related Rates WS #4   Name Key   Date____

1994 AB5, BC2

1) A circle is inscribed in a square as shown in the figure. The circumference of the circle is increasing at a constant rate of 6 inches per minute. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius r has circumference $C = 2\pi r$ and area $A = \pi r^2$.)



$\frac{dC}{dt} = 6$ in/min $\frac{dC}{dt} = 2\pi\left(\frac{dr}{dt}\right)$

a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

$P = 8r$

$\frac{dP}{dt} = 8\frac{dr}{dt}$

$6 = 2\pi\left(\frac{dr}{dt}\right)$

$\frac{6}{2\pi} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{\pi}$ in/min

$\frac{dP}{dt} = 8\left(\frac{3}{\pi}\right) = \boxed{\frac{24}{\pi} \text{ in/min}}$

A_e = Area enclosed
A_c = Area circle
A_s = Area square

b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

$A = 25\pi$

$\frac{dA_e}{dt} =$ ____

$A = \pi r^2$

$25\pi = \pi r^2$

$25 = r^2$

$\boxed{5 = r}$

$A_e = A_s - A_c$

$A_s = (2r)^2$

$A_c = \pi r^2$

$A_e = 4r^2 - \pi r^2$

$\frac{dA_e}{dt} = 8r\left(\frac{dr}{dt}\right) - 2\pi r\left(\frac{dr}{dt}\right)$

$\frac{dr}{dt} = \frac{3}{\pi}$ in/min

$\frac{dA}{dt} = 8(5)\left(\frac{3}{\pi}\right) - 2\pi(5)\left(\frac{3}{\pi}\right)$

$\boxed{\frac{dA}{dt} = \left(\frac{120}{\pi} - 30\right) \text{ in}^2\text{/min}}$

$V = 4(t)^{1/2}$

2. Suppose that a spherical balloon grows in such a way that after t seconds, $V = 4\sqrt{t}$ in^3. How fast is the radius changing after 64 seconds? ($V = \frac{4}{3}\pi r^3$)

$\longrightarrow 32 = \frac{4}{3}\pi(r^3)$ $\frac{24}{\pi} = r^3$

$r = \sqrt[3]{24/\pi}$

$t = 64$

$V = 4\sqrt{64} = 4 \cdot 8 = 32$ in^3

$\frac{dV}{dt} = 4 \cdot \frac{1}{2} t^{-1/2}\left(\frac{dt}{dt}\right)$

$\frac{dV}{dt} = \frac{2}{\sqrt{t}} = \frac{2}{\sqrt{64}} = \frac{2}{8} = \frac{1}{4}$ in^3/s

Find $\frac{dr}{dt} =$ ____

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

$\frac{1}{4} = 4\pi\left(\sqrt[3]{\frac{24}{\pi}}\right)^2 \frac{dr}{dt}$

$\frac{.1}{4 \cdot 4\pi\left(\sqrt[3]{\frac{24}{\pi}}\right)^2} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{194.974} \approx \boxed{0.005 \text{ in/s}}$

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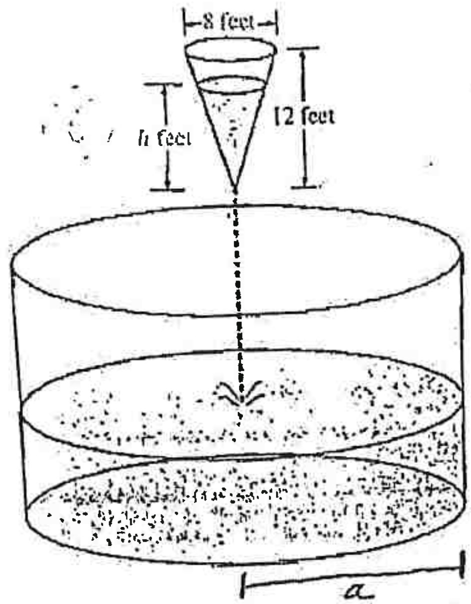
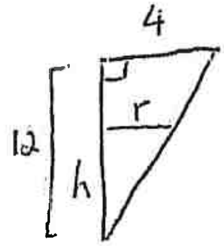
3. 1995 AB 5





$\frac{r}{4} = \frac{h}{12}$

$12r = 4h$

$r = \frac{4}{12}h$

$\boxed{r = \frac{h}{3}}$

$\frac{dh}{dt} = h - 12$ ft/min

As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

$V = \frac{\pi}{3}r^2 h$

$V = \frac{\pi}{3}\left(\frac{h}{3}\right)^2 h$

$V = \frac{\pi}{3} \cdot \frac{h^2}{9} \cdot h$

$\boxed{V = \frac{\pi}{27}h^3}$

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

$V = \frac{\pi}{27}h^3$

$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2 \left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{\pi}{9}h^2 \cdot \left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{\pi}{9}h^2 \cdot (h-12)$

$\frac{dV}{dt} = \frac{\pi}{9}(3)^2(3-12)$

$= \frac{\pi}{\cancel{9}} \cdot \cancel{9}(-9)$

$\boxed{\frac{dV}{dt} = -9\pi \text{ ft}^3/\text{min}}$

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure. $(V = \pi a^2 y)$

$\frac{dV}{dt} = 9\pi \text{ ft}^3/\text{min}$

$Area(base) = 400\pi$

$A = \pi r^2$

$400\pi = \pi r^2$

$400 = r^2$

$20 = r$

$r = 20$ ft

$a = 20$

$V = \pi(20)^2 y$

$V = 400\pi y$

$\frac{dV}{dt} = 400\pi\left(\frac{dy}{dt}\right)$

$9\pi = 400\pi\left(\frac{dy}{dt}\right)$

$\frac{9}{400} = \frac{dy}{dt}$

$\boxed{\frac{dy}{dt} = \frac{9}{400} \text{ ft/min}}$

Key (18)

Ch. 4.3 Related Rates Exercise Problems (Day 1)

Pg. 286-291 #9, 10, 22, 23, 35, 38

9. Volume of a Cube If each edge of a cube is increasing at the constant rate of 3 cm/s, how fast is the volume of the cube increasing when the length x of an edge is 10 cm?

$V = x^3$

$\frac{dV}{dt} = 3x^2\left(\frac{dx}{dt}\right)$

$\frac{dx}{dt} = 3\,cm/sec$

$x = 10cm$

$\frac{dV}{dt} = \_\_?$

$\frac{dV}{dt} = 3(10)^2(3)$

$\boxed{\frac{dV}{dt} = 900\ cm^3/sec}$

10. Volume of a Sphere If the radius of a sphere is increasing at 1 cm/s, find the rate of change of its volume when the radius is 6 cm.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 3\cdot\frac{4}{3}\pi r^2\left(\frac{dr}{dt}\right)$

$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

$\frac{dr}{dt} = 1\,cm/sec$

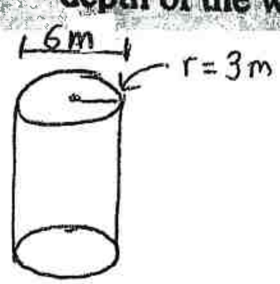
$r = 6cm$

$\frac{dV}{dt} = \_\_?$

$\frac{dV}{dt} = 4\pi(6)^2(1)$

$\boxed{\frac{dV}{dt} = 144\pi \approx 452.389\ cm^3/sec}$

22. Filling a Tank Water is flowing into a vertical cylindrical tank of diameter 6 m at the rate of 5 m³/min. Find the rate at which the depth of the water is rising.



$\frac{dV}{dt} = 5\,m^3/min$

$\frac{dh}{dt} = \_\_?$

$V = \pi r^2 h$

$V = \pi(3)^2 h$

$V = 9\pi h$

$\frac{dV}{dt} = 9\pi\left(\frac{dh}{dt}\right)$

$5 = 9\pi\left(\frac{dh}{dt}\right)$

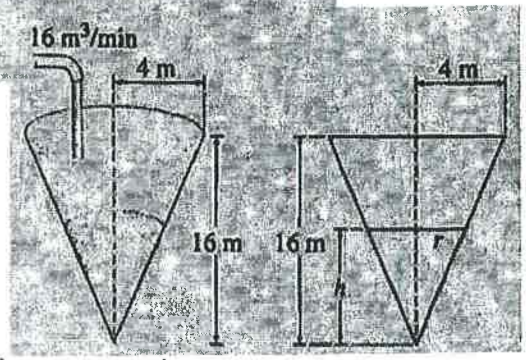
$\frac{5}{9\pi} = \frac{dh}{dt}$

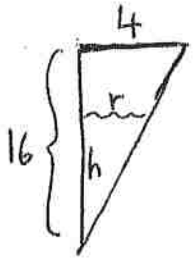
$\boxed{\frac{dh}{dt} = \frac{5}{9\pi}\ m/min \approx 0.177\ m/min}$

(19)

p.38

23. Fill Rate A container in the form of a right circular cone (vertex down) has radius 4 m and height 16 m. See the figure. If water is poured into the container at the constant rate of 16 m^3/min, how fast is the water level rising when the water is 8 m deep? *Hint:* The volume V of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.





$\frac{r}{4} = \frac{h}{16}$

$16r = 4h$

$r = \frac{4}{16}h = \frac{1}{4}h = \frac{h}{4}$

$V = \frac{\pi}{3}r^2 h$

$V = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h$

$V = \frac{\pi}{3}\cdot\frac{h^2}{16}\cdot h$

$V = \frac{\pi}{48}h^3$

$\frac{dV}{dt} = 3\cdot\frac{\pi}{48}h^2\left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{3\pi}{48}h^2\left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{\pi}{16}h^2\left(\frac{dh}{dt}\right)$

$16 = \frac{\pi}{16}(8)^2\left(\frac{dh}{dt}\right)$

Given: $\frac{dV}{dt} = 16\text{m}^3/\text{min}$

$\frac{dh}{dt} = \_\_?$ $h = 8$

$16 = \frac{\pi}{16}\cdot 64\left(\frac{dh}{dt}\right)$

$16\cdot\frac{16}{\pi\cdot 64} = \frac{dh}{dt}$

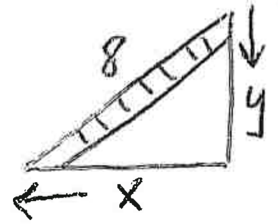
$\boxed{\frac{dh}{dt} = \frac{4}{\pi}\ \text{m/min}}$

35. Falling Ladder An 8-m ladder is leaning against a vertical wall. If a person pulls the base of the ladder away from the wall at the rate of 0.5 m/s, how fast is the top of the ladder moving down the wall when the base of the ladder is

(a) 3 m from the wall?

(b) 4 m from the wall?

(c) 6 m from the wall?



$x^2 + y^2 = 8^2$

$\boxed{2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 0}$

a) $x = 3$

$y = \sqrt{55}$

$\frac{dx}{dt} = 0.5$

$\frac{dy}{dt} = \_\_?$

$2(3)(0.5) + 2\sqrt{55}\left(\frac{dy}{dt}\right) = 0$

$\boxed{\frac{dy}{dt} = \frac{-1.5}{\sqrt{55}}\ \text{m/s}}$

b) $x = 4$ $y = 4\sqrt{3}$

$\frac{dx}{dt} = 0.5$ $\frac{dy}{dt} = \_\_?$

$2(4)(0.5) + 2(4\sqrt{3})\left(\frac{dy}{dt}\right) = 0$

$\boxed{\frac{dy}{dt} = \frac{-0.5}{\sqrt{3}}\ \text{m/s}}$

c) $x = 6$ $y = 2\sqrt{7}$

$dx/dt = 0.5$ $dy/dt = \_\_?$

$2(6)(0.5) + 2(2\sqrt{7})\left(\frac{dy}{dt}\right) = 0$

$\boxed{\frac{dy}{dt} = \frac{-3}{2\sqrt{7}}\ \text{m/s}}$

38. Tracking a Rocket When a rocket is launched, it is tracked by a tracking dish on the ground located a distance D from the point of launch. The dish points toward the rocket and adjusts its angle of elevation θ to the horizontal (ground level) as the rocket rises. Suppose a rocket rises vertically at a constant speed of 2.0 m/s, with the tracking dish located 150 m from the launch point. Find the rate of change of the angle θ of elevation of the tracking dish with respect to time t (tracking rate) for each of the following:

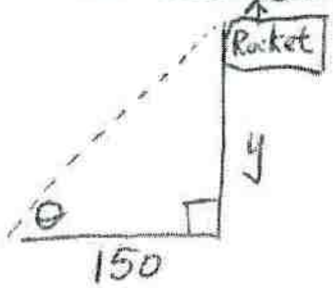
(a) Just after launch.

(b) When the rocket is 100 m above the ground.

(c) When the rocket is 1.0 km above the ground.

(d) Use the results in (a)–(c) to describe the behavior of the tracking rate as the rocket climbs higher and higher. What limit does the tracking rate approach as the rocket gets extremely high?

d) Based on the previous results, the tracking rate $\left(\frac{d\theta}{dt}\right)$ continues to decrease. As $\theta \to \frac{\pi}{2}$, $\sec\theta \to \infty$, so $\boxed{\frac{d\theta}{dt} \to 0}$



$\tan\theta = \frac{y}{150}$

$\tan\theta = \frac{1}{150}y$

$\boxed{\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{150}\left(\frac{dy}{dt}\right)}$

a) Just after launch:

$y = 0$

$\tan\theta = 0$

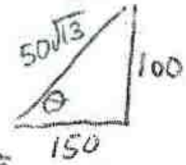
$\sec\theta = 1$

$\frac{dy}{dt} = 2\,m/sec$

$(1)^2\left(\frac{d\theta}{dt}\right) = \frac{1}{150}(2)$

$\boxed{\frac{d\theta}{dt} = \frac{1}{75}\ rad/sec.}$

b) $y = 100$



$100^2 + 150^2 = c^2$

$c = \sqrt{32500}$

$c = 50\sqrt{13}$

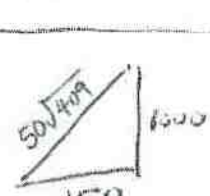
$\sec\theta = \frac{50\sqrt{13}}{150} = \frac{\sqrt{13}}{3}$

$\sec^2\theta = \left(\frac{\sqrt{13}}{3}\right)^2 = \frac{13}{9}$

$\left(\frac{13}{9}\right)\left(\frac{d\theta}{dt}\right) = \frac{1}{150}(2)$

$\boxed{\frac{d\theta}{dt} = \frac{1}{75}\cdot\frac{9}{13} = \frac{9}{975}\ rad/sec.\ \text{or}\ \frac{3}{325}\ rad/sec}$

c) $y = 1000\,m$



$1000^2 + 150^2 = c^2$

$c^2 = 1022500$

$c = 50\sqrt{409}$

$\left(\frac{50\sqrt{409}}{150}\right)^2\left(\frac{d\theta}{dt}\right) = \frac{1}{150}(2)$

$\left(\frac{409}{9}\right)\frac{d\theta}{dt} = \frac{1}{75}$

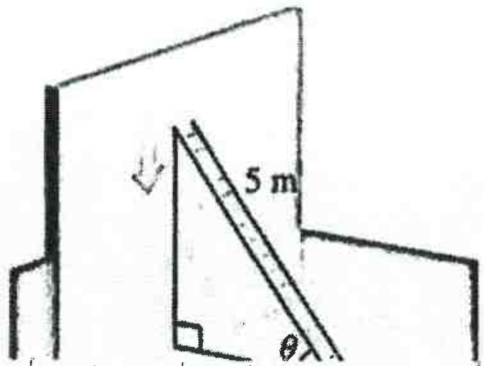
$\boxed{\frac{d\theta}{dt} = \frac{1}{75}\cdot\frac{9}{409} = \frac{9}{30675} \approx 0.000293\ rad/sec}$

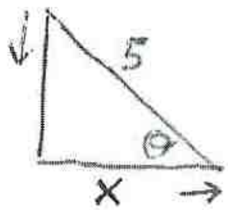
Key

Ch. 4.3 Related Rates Exercise Problems (Day 2)

Pg. 286-291 #19, 39, 40, 53, 54

19. Change in Inclination A ladder 5 m long is leaning against a wall. If the lower end of the ladder slides away from the wall at the rate of 0.5 m/s, at what rate is the inclination θ of the ladder with respect to the ground changing when the lower end is 4 meter from the wall. $\rightarrow 0.5$ m/sec.





$x = 4$

$\frac{dx}{dt} = 0.5$

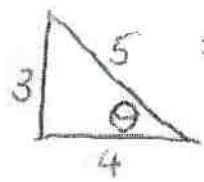
$\frac{d\theta}{dt} = \_\_?$

$\cos\theta = \frac{x}{5}$

$\cos\theta = \frac{1}{5}x$

$-\sin\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{5}\left(\frac{dx}{dt}\right)$

$-\left(\frac{3}{5}\right)\left(\frac{d\theta}{dt}\right) = \frac{1}{5}(0.5)$



$\sin\theta = \frac{3}{5}$

$\frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{1}{2}\cdot\frac{-5}{3}$

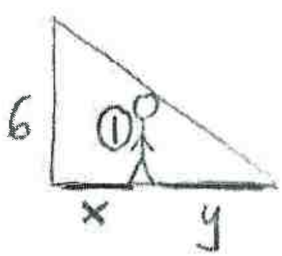
$\boxed{\frac{d\theta}{dt} = \frac{-1}{6}\text{ rad/sec}}$

39. Lengthening Shadow A child, 1 m tall, is walking directly under a street lamp that is 6 m above the ground. If the child walks away from the light at the rate of 20 m/min, how fast is the child's shadow lengthening?

RoC walking $\rightarrow \frac{dx}{dt}$

RoC shadow $\rightarrow \frac{dy}{dt}$

RoC tip of shadow $\rightarrow \frac{dx}{dt} + \frac{dy}{dt}$



$\frac{1}{6} = \frac{y}{x+y}$

$x + y = 6y$

$x = 5y$

$\frac{dx}{dt} = 5\left(\frac{dy}{dt}\right)$

$\frac{dx}{dt} = 20\text{ m/min}$

$\frac{dy}{dt} = \_\_?$

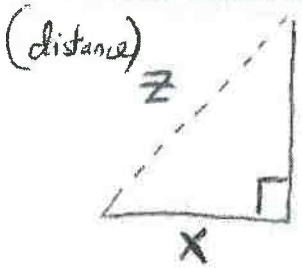
$20 = 5\left(\frac{dy}{dt}\right)$

$\frac{20}{5} = \frac{dy}{dt}$

$\boxed{\frac{dy}{dt} = 4\text{ m/min}}$

22

40. Approaching a Pole A boy is walking toward the base of a pole 20 m high at the rate of 4 km/h. At what rate (in meters per second) is the distance from his feet to the top of the pole changing when he is 5 m from the pole?



$x^2+20^2=z^2$

$2x\left(\frac{dx}{dt}\right)+0=2z\left(\frac{dz}{dt}\right)$

$x=5\,m$

$\frac{dx}{dt}=\frac{-4\ Km}{h}\cdot\frac{1000\,m}{Km}\cdot\frac{1h}{3600\,sec}$

$\frac{dx}{dt}=\frac{-10}{9}\ m/sec$

$z^2=(5)^2+20^2 \rightarrow z=5\sqrt{17}$

$2(5)\left(\frac{-10}{9}\right)=2(5\sqrt{17})\frac{dz}{dt}$

$\boxed{\frac{dz}{dt}=\frac{-10}{9\sqrt{17}}\approx -0.269\ m/sec}$

53. Change in Volume The height h and width x of an open box with a square base are related to its volume by the formula $V=hx^2$. Discuss how the volume changes

(a) if h decreases with time, but x remains constant.

(b) if both h and x change with time.

a) $\boxed{\frac{dV}{dt}=\frac{dh}{dt}x^2}$

b) $V=\underbrace{h}_{f}\underbrace{x^2}_{g}$

$\frac{dV}{dt}=\underbrace{\frac{dh}{dt}}_{f'}\cdot\underbrace{x^2}_{g}+\underbrace{h}_{f}\cdot\underbrace{2x\left(\frac{dx}{dt}\right)}_{g'}$

$\boxed{\frac{dV}{dt}=x^2\left(\frac{dh}{dt}\right)+2hx\left(\frac{dx}{dt}\right)}$

54. Rate of Change Let $y=2e^{\cos x}$. If both x and y vary with time in such a way that y increases at a steady rate of 5 units per second, at what rate is x changing when $x=\frac{\pi}{2}$?

$\frac{dy}{dt}=5$

$x=\pi/2$

$\frac{dx}{dt}=\_\_?$

$y=2e^{\cos x}$

$\frac{dy}{dt}=2e^{\cos x}\cdot -\sin x\left(\frac{dx}{dt}\right)$

$5=2e^{\cos(\pi/2)}\cdot -\sin(\pi/2)\left(\frac{dx}{dt}\right)$

$5=2e^0(-1)\left(\frac{dx}{dt}\right)$

$\frac{5}{-2}=\frac{dx}{dt}$

$\boxed{\frac{dx}{dt}=\frac{-5}{2}\ units/sec.}$

Key

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4.3 AP Practice Problems (p. 290-291)

1. A spherical balloon is inflated at the rate of 50 m^3/min. Find the rate at which the radius of the balloon is increasing when the diameter is 20 m.

(A) $\frac{1}{2\pi}$ m/min (B) $\frac{5}{8\pi}$ m/min

$\boxed{\text{(C)}}$ $\frac{1}{8\pi}$ m/min (D) $\frac{5}{4\pi}$ m/min

$\frac{dV}{dt} = 50\ m^3/min$

$\frac{dr}{dt} = \_\_\_?$

$d = 20$

$\to r = 10$

$V = \frac{4}{3}\pi r^3$ | $\frac{dV}{dt} = 3 \cdot \frac{4}{3}\pi r^2 \left(\frac{dr}{dt}\right)$ | $\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$

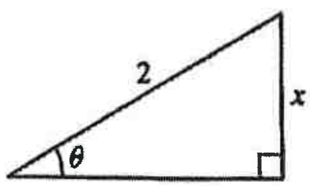
$\to 50 = 4\pi(10)^2\left(\frac{dr}{dt}\right)$

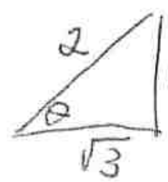
$\frac{50}{400\pi} = \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = \frac{1}{8\pi}\ m/min}$

2. In the right triangle below, θ is changing at the rate of 2 radians per second. At what rate is x changing at the instant when $x = 1$ cm?

$\to \frac{d\theta}{dt} = 2\ rad/sec$ $\quad \frac{dx}{dt} = \_\_\_?$





$\to \cos\theta = \frac{\sqrt{3}}{2}$

(A) 2 cm/s $\boxed{\text{(B)}}$ $2\sqrt{3}$ cm/s (C) $\sqrt{3}$ cm/s (D) $4\sqrt{3}$ cm/s

*create trig equation involving the given information:

$\sin\theta = \frac{x}{2}$ | $\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{2}\left(\frac{dx}{dt}\right)$

$\sin\theta = \frac{1}{2}x$

$\left(\frac{\sqrt{3}}{2}\right)(2) = \frac{1}{2}\left(\frac{dx}{dt}\right)$

$\frac{\sqrt{3}}{\cancel{2}} \cdot \cancel{2} \cdot \frac{2}{1} = \frac{dx}{dt}$

$\boxed{\frac{dx}{dt} = 2\sqrt{3}\ cm/sec}$

3. The radius of a circle is decreasing at a constant rate of 2 in./min. What is the rate of change in the area of the circle when its area is 25π in.^{2}?

$\boxed{\text{(A)}}$ -20π in.^{2}/min (B) -25π in.^{2}/min

(C) 20π in.^{2}/min (D) $20\pi^2$ in.^{2}/min

$A = \pi r^2$ | $25\pi = \pi r^2$ | $25 = r^2$ | $r = 5$

$A = \pi r^2$ | $\frac{dA}{dt} = \_\_\_?$ $\quad A = 25\pi$

$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$ $\to$ $\frac{dA}{dt} = 2\pi(5)(-2)$

$\frac{dr}{dt} = -2\ in/min$ | $\frac{dA}{dt} = -20\pi$

$\boxed{\frac{dA}{dt} = -20\pi\ in^2/min}$

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4. The radius r of a sphere is increasing at a rate of 2 cm/s. At the instant when $r = 12$ cm, what is the rate of change in the surface area S of the sphere? (The surface area S of a sphere with radius r is $S = 4\pi r^2$.)

(A) 96π cm^2/s (B) 1152π cm^2/s

(C) 576π cm^2/s $\boxed{\text{(D)}}$ 192π cm^2/s

$S = 4\pi r^2$

$\frac{dS}{dt} = 2 \cdot 4\pi r\left(\frac{dr}{dt}\right)$

$\frac{dS}{dt} = 8\pi r\left(\frac{dr}{dt}\right)$

$\frac{dr}{dt} = 2$ cm/sec

$r = 12$

$\frac{dS}{dt} = \_\_\_ ?$

$\frac{dS}{dt} = 8\pi(12)(2)$

$\boxed{\frac{dS}{dt} = 192\pi \text{ cm}^2/\text{sec}}$

5. The sides of the rectangle shown below are increasing so that the rate of change of y with respect to time t is three times the rate of change of x with respect to t. If $\frac{dc}{dt} = 1$, what is the rate of change of x when $x = 6$ and $y = 8$?

$\frac{dy}{dt} = 3\left(\frac{dx}{dt}\right)$

$\frac{dc}{dt} = 1$

$\frac{dx}{dt} = \_\_\_ ?$

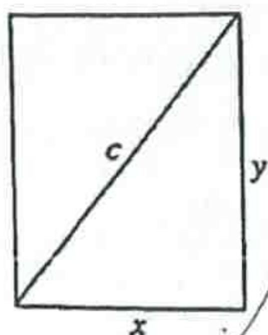
$x = 6$

$y = 8$

$c = 10$

(A) 3 $\boxed{\text{(B)}}$ $\frac{1}{3}$

(C) 1 (D) $\frac{1}{6}$



$x^2 + y^2 = c^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2c\left(\frac{dc}{dt}\right)$

$2(6)\left(\frac{dx}{dt}\right) + 2(8) \cdot 3\left(\frac{dx}{dt}\right) = 2(10)(1)$

$12\left(\frac{dx}{dt}\right) + 48\left(\frac{dx}{dt}\right) = 20$

$60\left(\frac{dx}{dt}\right) = 20$

$\boxed{\frac{dx}{dt} = \frac{20}{60} = \frac{1}{3}}$

6. The area of a circle is increasing at a rate of 48π ft^2/h. How fast is the radius of the circle increasing when its area is 36π ft^2?

$\boxed{\text{(A)}}$ 4 ft/h (B) 6 ft/h (C) $4\sqrt{3}$ ft/h (D) $\frac{4}{\pi}$ ft/h

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$

$\frac{dA}{dt} = 48\pi$ ft^2/h

$\frac{dr}{dt} = \_\_\_ ?$

$A = 36\pi$

$A = \pi r^2$

$36\pi = \pi r^2$

$36 = r^2$

$6 = r$

$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$

$48\pi = 2\pi(6)\left(\frac{dr}{dt}\right)$

$\frac{48\pi}{12\pi} = \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = 4 \text{ ft/h}}$

7. The radius r and height h of a right circular cone are both increasing at a constant rate of 2 cm/h. At what rate in centimeters cubed per hour is the volume V of the cone increasing when $r = 6$ cm and $h = 15$ cm? (The volume V of a right circular cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)

$\frac{dr}{dt} = 2\text{cm/h}$

$\frac{dh}{dt} = 2\text{cm/h}$

$r = 6\text{cm}$

$h = 15\text{cm}$

(A) 24π cm³/h (B) 96π cm³/h

$\boxed{\text{(C)}}$ 144π cm³/h (D) 180π cm³/h

$$V = \overbrace{\frac{\pi}{3}r^2}^{f}\overbrace{h}^{g}$$

$$\frac{dV}{dt} = \overbrace{2\cdot\frac{\pi}{3}r\left(\frac{dr}{dt}\right)}^{f'}\cdot\overbrace{h}^{g} + \overbrace{\frac{\pi}{3}r^2}^{f}\cdot\overbrace{\left(\frac{dh}{dt}\right)}^{g'}$$

$$\frac{dV}{dt} = \frac{2\pi}{3}(6)(2)(15) + \frac{\pi}{3}(6)^2(2)$$

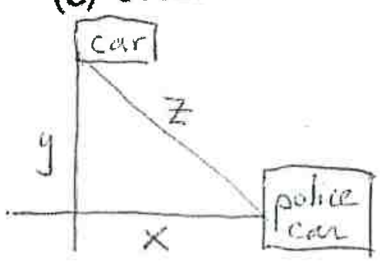
$$\frac{dV}{dt} = 120\pi + 24\pi$$

$$\boxed{\frac{dV}{dt} = 144\pi \text{ cm}^3/\text{h}}$$

8. Two roads cross at right angles. A police officer sits in a car 65 m east of the crossing and observes a car speeding northbound at 84 m/s. At what speed (in meters per second) is the car distancing itself from the police officer 5 seconds after it passes the crossing?

(A) 166.024 m/s $\boxed{\text{(B)}}$ 83.012 m/s

(C) 84 m/s (D) 95.859 m/s



$x = 65$ $\frac{dx}{dt} = \underline{0}$

$y = 5(84) = 420$ $\frac{dy}{dt} = 84$

$z = \underline{425}$ $\frac{dz}{dt} = \underline{\quad}$?

$$x^2 + y^2 = z^2$$

$$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$$

$$\cancel{2(65)(0)} + 2(420)(84) = 2(425)\left(\frac{dz}{dt}\right)$$

$$70560 = 850\left(\frac{dz}{dt}\right)$$

$$\boxed{\frac{dz}{dt} = \frac{70560}{850} \approx 83.012 \text{ m/s}}$$

(26)

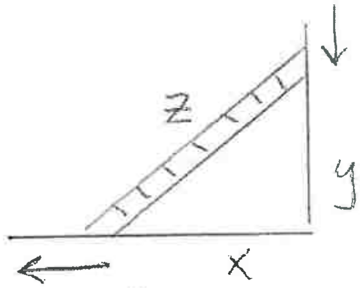
9. A roofer's 13-meter ladder is placed against the wall of a building with its base on level ground. The top of the ladder slips down the wall as the bottom of the ladder slips away from the building at a constant rate of 5 m/s.

a) $-12\,m/s$
b) $-59.5\,m^2/sec$
c) -1 rad/sec.

(a) At what rate is the top of the ladder moving when it is 5 meters from the ground?

(b) At what rate is the area of the triangle formed by the ladder, the wall, and the ground changing when the top of the ladder is 5 m from the ground?

(c) If θ is the angle formed by the ladder and the ground, what is the rate of change in θ when the top of the ladder is 5 m from the ground?



$x = \underline{12}$ $\quad dx/dt = +5$ m/sec

$y = 5$ $\quad dy/dt = \_\_\_ ?$

$z = 13$ $\quad dz/dt = 0$

$$x^2 + y^2 = z^2$$

$$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$$

$$2(12)(5) + 2(5)\left(\frac{dy}{dt}\right) = 2(13)(0)$$

$$120 + 10\left(\frac{dy}{dt}\right) = 0$$

$$10\left(\frac{dy}{dt}\right) = -120$$

a) $\boxed{\frac{dy}{dt} = -12\,m/s}$

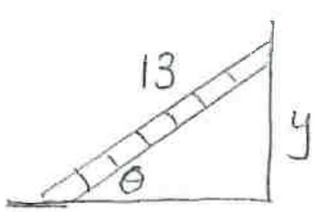
b) $\text{Area} = \frac{1}{2}xy$ $\quad$ *Area of triangle is $A = \frac{1}{2}bh$

$$\frac{dA}{dt} = \underbrace{\frac{1}{2}\left(\frac{dx}{dt}\right)}_{f'} \cdot \underbrace{y}_{g} + \underbrace{\frac{1}{2}x}_{f} \cdot \underbrace{\left(\frac{dy}{dt}\right)}_{g'}$$

$$\frac{dA}{dt} = \frac{1}{2}(5)(5) + \frac{1}{2}(12)(-12)$$

$$\boxed{\frac{dA}{dt} = \frac{25}{2} - \frac{144}{2} \rightarrow \frac{-119}{2} \text{ or } -59.5\,m^2/sec}$$

c)



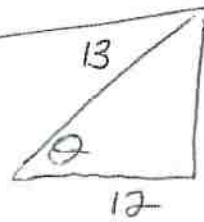
$$\sin\theta = \frac{y}{13}$$

$$\sin\theta = \frac{1}{13}y$$

$$\boxed{\cos\theta}\left(\frac{d\theta}{dt}\right) = \frac{1}{13}\left(\frac{dy}{dt}\right)$$

$$\boxed{\left(\frac{12}{13}\right)}\left(\frac{d\theta}{dt}\right) = \frac{1}{13}(-12)$$

$$\frac{d\theta}{dt} = \frac{13}{12} \cdot \frac{-12}{13} = -1$$



$$\rightarrow \boxed{\cos\theta = \frac{12}{13}}$$

$$\boxed{\frac{d\theta}{dt} = -1\text{ rad/sec}}$$

Key 27

AP Calculus – 4.4 Notes - L'Hopital's Rule and Indeterminate Form

Recall: When evaluating limits, $\boxed{\text{first try direct substitution!}}$ $\lim\limits_{x\to 3}\frac{2x-5}{x} = \frac{6-5}{3} = \frac{1}{3}$

1. $\lim\limits_{x\to 2}\frac{x^2-7x+10}{x-2} = \frac{4-14+10}{2-2} \to \frac{0}{0} \to \lim\limits_{x\to 2}\frac{(x-5)\cancel{(x-2)}}{\cancel{(x-2)}} \to 2-5 = \boxed{-3}$

$\lim\limits_{x\to 2}\frac{2x-7}{1} \to 4-7 = \boxed{-3}$

Indeterminate Form

L'Hospital's Rule:

Suppose $f(a)=0$ and $g(a)=0$ and $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$. L'Hopital's Rule allows you to apply the following:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \to \frac{f'(a)}{g'(a)}$$

Evaluate each limit. Use L'Hospital's when possible.

2. $\lim\limits_{x\to 2}\frac{x-2}{3x^3-6x^2+x-2} \to \frac{0}{0}$

$\lim\limits_{x\to 2}\frac{1}{9x^2-12x+1} \to \frac{1}{9(4)-24+1} \to \frac{1}{36-24+1} \to \boxed{\frac{1}{13}}$

3. $\lim\limits_{x\to 0}\frac{\sin(6x)}{x} \to \frac{0}{0}$

$\lim\limits_{x\to 0}\frac{\cos(6x)\cdot 6}{1} \to \frac{6\cos(0)}{1} \to 6(1) = \boxed{6}$

4. $\lim\limits_{x\to 0}\frac{1-\cos(x)}{x^2} \to \frac{0}{0}$

$\lim\limits_{x\to 0}\frac{-(-\sin x)}{2x} \to \frac{\sin 0}{2(0)} \to \frac{0}{0}$

$\lim\limits_{x\to 0}\frac{\cos x}{2} \to \frac{\cos 0}{2} = \boxed{\frac{1}{2}}$

5. $\lim\limits_{x\to\infty}\frac{2x^2}{e^{2x}} \to \frac{\infty}{\infty}$

$\lim\limits_{x\to\infty}\frac{4x}{2e^{2x}} \to \frac{\infty}{\infty}$

$\lim\limits_{x\to\infty}\frac{4}{4e^{2x}} \to \frac{0}{\infty} = \boxed{0}$

* OR Apply comparative growth rate

$L < R < P < E$

$\lim\limits_{x\to\infty}\frac{\text{slower}}{\text{faster}} = \boxed{0}$

L'HOSPITAL'S IS NOT THE QUOTIENT RULE!!

6. $\frac{d}{dx}\frac{\sin(6x)}{x}$

$$f'(x) = \frac{\underbrace{\cos(6x)\cdot 6}_{f'}\,\underbrace{x}_{g} - \underbrace{\sin(6x)}_{f}\cdot\underbrace{1}_{g'}}{\underbrace{x^2}_{g^2}} \to \boxed{\frac{6x\cos(6x)-\sin(6x)}{x^2}}$$

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Practice Problems:

Find the following. Use L'Hôpital's when possible.

1. $\lim_{x\to 1}\frac{x-1}{x^2-3x+2} \to \frac{0}{0}$ $\lim_{x\to 1}\frac{1}{2x-3} = \frac{1}{-1} = \boxed{-1}$	2. $\lim_{x\to -5}\frac{x^2-2x-35}{x+5} \to \frac{0}{0}$ $\lim_{x\to -5}\frac{2x-2}{1} \to \boxed{-12}$ $-5(2)-2$	3. $\lim_{x\to 0}\frac{4x}{\ln(x+1)} \to \frac{0}{0}$ $\lim_{x\to 0}\frac{4}{\frac{1}{x+1}} \to \frac{4}{\frac{1}{1}} = \boxed{4}$
4. $\lim_{x\to 0}\frac{x-1}{x^2-3x+2} \to \boxed{\frac{-1}{2}}$	5. $\lim_{x\to 1}\frac{2(x^2-1)}{\ln x^2} \to \frac{0}{0}$ $\lim_{x\to 1}\frac{2(2x)}{2(\frac{1}{x})} \to \frac{4}{2} = \boxed{2}$	6. $\frac{d}{dx}\frac{6x^2+x}{\sin(x)}$ Quotiet Rule $\boxed{\frac{(12x+1)\sin x-(6x^2+x)\cos x}{\sin^2 x}}$

16. If $f(x) = 2x^3 + 5$, then $\lim_{x\to 0}\frac{f(x)-f(0)}{x^3}$ is $\to \lim_{x\to 0}\frac{2x^3+5-5}{x^3} \to \lim_{x\to 0}\frac{2x^3}{x^3} \to \frac{0}{0}$

L'Hopital's $\to \lim_{x\to 0}\frac{f'(x)}{3x^2} \to \frac{6x^2}{3x^2} \to \boxed{2}$

(A) 0 (B) 1 (C) 2 (D) 3 (E) The limit does not exist.

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17. Functions f, g, and h are twice-differentiable functions with $g(3) = h(3) = 5$. The line $y = 5 + \frac{1}{2}(x - 3)$ is tangent to both the graph of g at $x = 3$ and the graph of h at $x = 3$.

a. Find $h'(3)$.

$h(x)$ is tangent to $y = \frac{1}{2}(x-3)+5$

* shares same slope, so $h'(3) = \frac{1}{2}$

b. Let a be the function given by $a(x) = \underbrace{2x^3}_{f}\underbrace{h(x)}_{g}$. Write an expression for $a'(x)$. Find $a'(3)$.

$$a'(x) = \underbrace{6x^2}_{f'} \cdot \underbrace{h(x)}_{g} + \underbrace{2x^3}_{f} \cdot \underbrace{h'(x)}_{g'}$$

$$a'(3) = 6(3)^2 \cdot h(3) + 2(3)^3 \cdot h'(3)$$

$$a'(3) = 6(9)(5) + 2(27)\left(\tfrac{1}{2}\right)$$

$$\boxed{a'(3) = 297}$$

c. The function h satisfies $h(x) = \frac{x^2-9}{1-(f(x))^3}$ for $x \neq 3$. It is known that $\lim_{x \to 3} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 3} h(x) = 5$ to find $f(3)$ and $f'(3)$. Show the work that leads to your answers.

$$\lim_{x \to 3} \frac{x^2-9}{1-[f(x)]^3} \rightarrow \frac{0}{0} \rightarrow \text{L'Hospital's} \rightarrow \lim_{x \to 3} \frac{2x}{-3[f(x)]^2 \cdot f'(x)}.$$

Chain Rule

out: $-[\ ]^3$

in: $f(x)$

$1-[f(3)]^3 = 0$

$(f(3))^3 = 1$

$f(3) = 1$

$$\lim_{x \to 3} \frac{2(3)}{-3[f(3)]^2 f'(3)} = 5$$

$$\frac{6}{-3[1]^2 \cdot f'(3)} = \frac{5}{1}$$

$$-3 \cdot 5 \cdot f'(3) = 6$$

$$f'(3) = \frac{6}{-15}$$

$$\boxed{f'(3) = \frac{-2}{5}}$$

4.4 AP Practice Problems (p. 301)

*L'Hopital's Rule

Key

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1. $\lim_{x\to 0} \frac{e^{4x}-1}{\sin(2x)} =$ $\frac{0}{0}$

(A) 0 $\boxed{\text{(B)}}$ 2 (C) 4 (D) does not exist

(L'H) $\lim_{x\to 0} \frac{e^{4x}(4)-0}{\cos(2x)\cdot 2} \to \frac{4e^0}{2\cos(0)} \to \frac{4(1)}{2(1)} \to \boxed{2}$

*$\sin 2x = 2\sin x\cos x$

2. $\lim_{x\to 0} \frac{1-\cos^2(3x)}{x^2} =$

(A) 18 $\boxed{\text{(B)}}$ 9 (C) 0 (D) 3

*$1-\cos^2\theta = \sin^2\theta$

$\lim_{x\to 0} \frac{\sin^2(3x)}{x^2} \to \frac{0}{0}$

$\lim_{x\to 0} \frac{[\sin(3x)]^2}{x^2}$

$\lim_{x\to 0} \frac{2[\sin(3x)]\cdot\cos(3x)\cdot 3}{2x}$

$\to \frac{2\sin(0)\cos(0)\cdot 3}{0} \to \frac{0}{0}$

$\lim_{x\to 0} \frac{6\sin(3x)\cos(3x)}{2x} \to \frac{3\cdot\sin(2\cdot 3x)}{2x}$

$\lim_{x\to 0} \frac{3\cdot\cos(6x)\cdot 6}{2} \to \frac{18(1)}{2} = \boxed{9}$

3. Find $\lim_{x\to\infty} \frac{x^{-3/2}}{\sin\frac{1}{x}}$.

(A) $\frac{3}{2}$ (B) 1 $\boxed{\text{(C)}}$ 0 (D) ∞

$\lim_{x\to\infty} \frac{1}{x^{3/2}\sin(\frac{1}{x})} = \boxed{0}$

4. $\lim_{x\to 1} \frac{\ln x^3}{x^2-1} =$

(A) 0 (B) 1 $\boxed{\text{(C)}}$ $\frac{3}{2}$ (D) 3

$\lim_{x\to 1} \frac{3\ln x}{x^2-1} \to \frac{0}{0} \to \lim_{x\to 1} \frac{\frac{3}{x}}{2x} \to \frac{\frac{3}{1}}{2} = \boxed{\frac{3}{2}}$

5. For any positive integer k, $\lim_{x\to\infty} \frac{\ln x}{x^k} =$

$\boxed{\text{(A)}}$ 0 (B) 1 (C) $k+1$ (D) ∞

* comparative growth Rates $L < R < P < E$

$\lim_{x\to\infty} \frac{\text{logs}}{\text{polynomial}} \to \boxed{0}$

6. $\lim_{\theta\to 0} \frac{1-\cos(2\theta)}{3\sin\theta} =$

(A) -2 (B) $\frac{2}{3}$ $\boxed{\text{(C)}}$ 0 (D) $-\frac{1}{3}$

$\lim_{\theta\to 0} \frac{1-\cos(2\theta)}{3\sin\theta} \to \frac{0}{0} \to \lim_{\theta\to 0} \frac{-\sin(2\theta)\cdot 2}{3\cos\theta} \to \frac{0}{3} \to \boxed{0}$

7. $\lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\cos x)}{\tan x} =$

(A) $-\infty$ $\boxed{\text{(B)}}$ 0 (C) 1 (D) ∞

$\lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\cos x)}{\tan x} \to \frac{\infty}{\infty} \to \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{-\sin x}{\cos x}}{\sec^2 x} \to$

$\lim_{x\to\pi/2^-} \frac{-\sin x}{\cos x}\cdot\frac{1}{\cos^2 x} \to \frac{-\sin x}{\cos^3 x}$ $\frac{1}{0.001}$

$\to \boxed{0}$

8. Which of the following are indeterminate forms at 0?

$\boxed{\text{I}}$ $\frac{x}{\ln(x+1)} \to \frac{0}{0}$ II $\frac{e^x}{x^2-2x} \to$ $\boxed{\text{III}}$ $\frac{x}{1-\cos(\pi x)} \to \frac{0}{0}$

(A) I only $\boxed{\text{(B)}}$ I and III only

(C) II and III only (D) I, II, and III

II. $\lim_{x\to\infty} \frac{e^x}{x^2-2x} \to \boxed{\infty}$ X

Key 33

Ch. 4 Review AP Practice Problems (p. 304)

1. If $y = \tan(3x + 2y)$, find the rate of change of y with respect to x at the origin.

$\boxed{\text{(A)}}$ -3 (B) 1 (C) 3 (D) 5

$y = \tan(3x+2y)$

$\frac{dy}{dx} = \sec^2(3x+2y) \cdot \left[3 + 2\left(\frac{dy}{dx}\right)\right]$

$\left.\frac{dy}{dx}\right|_{(0,0)} = \sec^2(0+0)\left[3 + 2\left(\frac{dy}{dx}\right)\right]$

$\frac{dy}{dx} = (1)\left(3 + 2\left(\frac{dy}{dx}\right)\right)$

$1\left(\frac{dy}{dx}\right) = 3 + 2\left(\frac{dy}{dx}\right)$

$-3 = \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = -3}$

2. The linear approximation for $f(x) = xe^x$ near $x = 2$ is

(A) $L(x) = 3e^2(x-2)$ (B) $L(x) = xe^x + xe^x(x-2)$

$\boxed{\text{(C)}}$ $L(x) = 2e^2 + 3e^2(x-2)$ (D) $L(x) = 2e^2 + 2e^2(x-2)$

$f(2) = 2e^2$

$f'(x) = \overbrace{1}^{f'} \cdot \overbrace{e^x}^{g} + \overbrace{x}^{f} \cdot \overbrace{e^x(1)}^{g'}$

$f'(2) = e^2 + 2e^2$

$f'(2) = 3e^2$

point: $(2, 2e^2)$

slope: $m = 3e^2$

$y - y_1 = m(x - x_1)$

$y - 2e^2 = 3e^2(x-2)$

$\boxed{y = 2e^2 + 3e^2(x-2)}$

3. $\lim_{x\to\infty} \frac{x}{\ln x} =$

(A) 0 (B) 1 (C) e $\boxed{\text{(D)}}$ ∞

* comparative growth rate L < R < P < E

$\lim_{x\to\infty} \frac{\text{polynomial}}{\text{logarithm}} \to \boxed{+\infty}$

$\lim_{x\to\infty} \frac{x}{\ln x}$

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4. The radius of a circle is increasing at a constant positive rate with respect to time. What is the radius of the circle when the rate of change of the area with respect to time is equal to twice the rate of change of the circumference with respect to time?

(A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 4

$\frac{dA}{dt} = 2\left(\frac{dC}{dt}\right)$

$A = \pi r^2$ | $C = 2\pi r$

$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$ | $\frac{dC}{dt} = 2\pi\left(\frac{dr}{dt}\right)$

$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right)$

$2\left(\cancel{\frac{dC}{dt}}\right) = r \cdot \left(\cancel{\frac{dC}{dt}}\right)$

$\boxed{2 = r}$

5. $\lim_{x \to 0} \frac{xe^x}{\sin x} =$

(A) 0 (B) 1 (C) e (D) ∞

$\lim_{x \to 0} \frac{xe^x}{\sin x} \to \frac{0}{0} \to \lim_{x \to 0} \frac{\overbrace{1}^{f'} \cdot \overbrace{e^x}^{g} + \overbrace{x}^{f} \cdot \overbrace{e^x}^{g'}}{\cos x} \to \frac{1e^0 + 0e^0}{\cos 0} \to \frac{1}{1} \to \boxed{1}$

6. Use the linear approximation to $f(x) = \tan x$ at $x = 0$ to approximate $f(0.2)$.

(A) -0.2 (B) 0 (C) 0.2 (D) 0.8

$f(0) = \tan(0) = 0$

$f'(x) = \sec^2 x$

$f'(0) = \sec^2 0 \to 1$

point: $(0, 0)$

slope: $m = 1$

$y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 0)$

$y = 1x$

$y(0.2) = 1(0.2) = \boxed{0.2}$

Free Response Questions

7. Water is being pumped into a cylindrical tank that measures 20 m in height and 4 m in radius at a constant rate of 5 m^3/h. Ten meters from the bottom of the tank there is a hole in the tank. Water leaks from that hole at a rate of 1 m^3/h.

   (a) Find the rate at which the water is rising in the tank until it reaches the leak.

   (b) Find the rate at which the water is rising in the tank after it passes the leak.

   (c) What is the total time it will take for the tank to begin to overflow?

a) $\frac{dh}{dt} = \frac{5}{16\pi}$ m/h

b) $\frac{dh}{dt} = \frac{1}{4\pi}$ m/h

c) 226.195 h.

$V = \pi r^2 h$

*radius is constant

$\frac{dV}{dt} = 5\, m^3/hr$

$\frac{dV}{dt}$ (hole) $= -1\, m^3/hr$

$V = \pi r^2 h$

$V = \pi (4)^2 h$

$V = 16\pi h$

$\frac{dV}{dt} = 16\pi \left(\frac{dh}{dt}\right)$

a) $5 = 16\pi \left(\frac{dh}{dt}\right)$

$\frac{5}{16\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{5}{16\pi}\ m/h}$

b) $\frac{dV}{dt} = 5 - 1 = 4\, m^3/hr$

$\frac{dV}{dt} = 16\pi \left(\frac{dh}{dt}\right)$

$4 = 16\pi \left(\frac{dh}{dt}\right)$

$\frac{4}{16\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{1}{4\pi}\ m/h}$

c) $\frac{1}{4\pi} \cdot t_1 = 10m$

$t_1 = 10 \cdot 4\pi = 40\pi$

$t_1 = \underline{125.664 hr.}$

(time taken to fill up first 10 meters of tank)

$\frac{5}{16\pi} \cdot t_2 = 10m$

$t_2 = 10 \cdot \frac{16\pi}{5} = \underline{100.5312 \text{ hrs.}}$

(time taken to fill up upper half of tank)

Total time: $125.664 + 100.5312 = \boxed{226.195 \text{ hrs.}}$

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8. The position function of an object moving along the x-axis is $s(t) = t \sin t + 3$, where s in meters and $t \geq 0$ in minutes.

(a) Find the initial position of the object. Find its position at $t = \frac{2\pi}{3}$ min.

(b) Find and interpret $s'\left(\frac{2\pi}{3}\right)$ in the context of this problem.

(c) Find the average velocity of the object from $t = 0$ to $t = \frac{2\pi}{3}$.

a) $4.814m \left(\frac{\pi\sqrt{3}+9}{3}\right)$

b) $\frac{3\sqrt{3}-2\pi}{6}$.

c) $\frac{\sqrt{3}}{2}$

a) $s(t) = t\sin t + 3$

$s(0) = 0\sin 0 + 3 = \boxed{3}$

$s\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3}\sin\left(\frac{2\pi}{3}\right) + 3 = \frac{2\pi}{3}\left(\frac{\sqrt{3}}{2}\right) + 3 \rightarrow \boxed{\frac{\pi\sqrt{3}+9}{3} \approx 4.814m}$

b) $s'(t) = \overbrace{1}^{f'} \cdot \overbrace{\sin(t)}^{g} + \overbrace{t}^{f} \cdot \overbrace{\cos(t)}^{g'} + 0$

$s'\left(\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) + \frac{2\pi}{3}\cos\left(\frac{2\pi}{3}\right) \rightarrow \frac{\sqrt{3}}{2} + \frac{2\pi}{3}\cdot\frac{-1}{2} \rightarrow \frac{\sqrt{3}}{2} - \frac{\pi}{3}$

$s'\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}-2\pi}{6} \approx -0.181$

The object is moving in the negative direction at 0.181 m/min at $t = \frac{2\pi}{3}$

c) Avg. velocity $= \frac{s\left(\frac{2\pi}{3}\right) - s(0)}{\frac{2\pi}{3} - 0}$

*Avg. velocity is $\frac{\text{change in position}}{\text{change in time}}$

$= \frac{\frac{\pi\sqrt{3}}{3} + 3 - 3}{\frac{2\pi}{3} - 0} \rightarrow \frac{\frac{\pi\sqrt{3}}{3}}{\frac{2\pi}{3}} \rightarrow \frac{\pi\sqrt{3}}{3}\cdot\frac{3}{2\pi} \rightarrow \boxed{\frac{\sqrt{3}}{2} \approx 0.866 \text{ m/min}}$